Journal of Mathematical Sciences: Advances and Applications
Volume 1, Number 2, 2008, Pages 423-430

A REMARK ON THE PRODUCTS OF DISTRIBUTIONS AND SEMIGROUPS

ADEM KILIÇMAN

Department of Mathematics
University Malaysia Terengganu
21030 Kuala Terengganu
Terengganu, Malaysia
e-mail: akilicman@umt.edu.my

Abstract

In this study we consider $\mathcal{D}$ the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ the space of distributions defined on $\mathcal{D}$. Now let $f(x, r)$ be distribution in $\mathcal{D}'$ and let $f(x, r)_n = f(x, r) * \delta_n(x, r)$ where δ_n is a certain sequence which converges to the Dirac-delta function. Then the products $f(x, r) \cdot [f(x, s)], [f(x, r)] \cdot [f(x, s)]$ are defined as the limit of the sequences $\{f(x, r)f(x, s)_n\}$, $\{f(x, r)_n f(x, s)_n\}$ provided that the limits h_1, h_2 exist in the sense of

$$\lim_{n \to \infty} \langle f(x, r)f(x, s)_n, \phi \rangle = \langle h_1, \phi \rangle,$$

$$\lim_{n \to \infty} \langle f(x, r)_n f(x, s)_n, \phi \rangle = \langle h_2, \phi \rangle,$$

respectively, for all ϕ in $\mathcal{D}$. In general, two products do not necessarily be equal. In this work, it is proved that two products are equal if they satisfy a property which we call semigroup condition. It is also proved that if products satisfy the semigroup condition then hold the associativity.

2000 Mathematics Subject Classification: 46F10.

Keywords and phrases: distributions, delta-function, regular sequence, product of distributions, semigroup.

Received July 13, 2008

1. Introduction

In mathematical analysis, distributions also known as generalized functions are objects which generalize functions and probability distributions.

They allow to extend the concept of derivative to all continuous functions and further they can be used to formulate generalized solutions of partial differential equations.

The theory of distributions is considered important and essential progress in the theory of partial differential equations as well as in mathematical physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution. However, in general multiplication of distributions is not possible for arbitrary distributions, see [15]. In fact this is considered an obstacle in the use of distributions and therefore several different definitions were developed in order to tackle the multiplication problem. There are also some objects such as $H(x)\delta(x)$, $x^{-1}\delta(x)$, $\delta(x)\delta(x)$, $\delta'\delta$, where δ is the Dirac function, are of special interest in physics and widely used in quantum theory, see [17]. For example, in physics, product of distributions such as $H\delta$ or δ^2 can be interpreted in several different ways; see Colombeau [1]. Thus in the literature, various definitions have been proposed just for δ^2 ranging from 0, $c\delta$, cx^{-2}, $c\delta + \frac{1}{2\pi i}\delta'$, to $c\delta + c'\delta'$ with arbitrary constants c, c'; see Oberguggenberger [14].

In this study, we prove that two products are equal if they satisfy a property what we call semigroup condition. It is also proved that if products satisfy the semigroup condition then they satisfy the associativity property.

Although it seems difficult to multiply arbitrary distributions, it is usually possible to define the product of a distribution f and an infinitely differential function g and this is given in the next definition.

Definition 1. Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. Then product $fg = gf$ is defined by

$$\langle fg, \phi \rangle = \langle gf, \phi \rangle = \langle f, g\phi \rangle$$

for all ϕ in $\mathcal{D}$.

It then follows easily by induction that

$$f^{(r)}g = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[Fg^{(i)} \right]^{(r-i)}$$

where

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}$$

for $r = 1, 2, \ldots$.

Let $f, g \in \mathcal{D}'$. Then there are two different approaches for defining the product of distributions: one it is known as regularization and passage to the limit, see Hirata and Ogata [6], Itano [8], Mikusiński [13], and the other way is Fourier Transform product, given two distributions $f, g \in \mathcal{D}'$ assume that their Fourier transforms $\mathcal{F}(f), \mathcal{F}(g)$ exist respectively then "Fourier product" is defined $f \cdot g = \mathcal{F}^{-1}(\mathcal{F} * \mathcal{F}g)$; see Hörmander [7]. The basis for all these definitions is product of a distribution and infinitely differential functions that was introduced by Schwartz [15] commutatively as the element $f\phi = \phi f \in \mathcal{D}'$. Later in [16], Temple followed the general idea of Mikusiński and Sikorski in [12], and developed a sequential theory of distributions which are defined as "regular sequences" of arbitrary functions then regular sequences can be used to define product of distributions and special functions, see [3].

For our next definition we let $\rho(x)$ be a fixed infinitely differentiable function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

The function δ_n is then defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It follows that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function δ. If now f is an arbitrary distribution in $\mathcal{D}'$, the function f_n is defined by

$$f(x, r)_n = (f * \delta_n)(t) = \langle f(t, r), \delta_n(x - t)\rangle.$$

where $n = 1, 2, \ldots$ and r is a fixed parameter. It follows that $\{f(x, r)_n\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x, r)$.

The following definition was proposed by Fisher is a generalization of the definition 1, see [2].

Definition 2. Let $f(x, r)$ and $g(x, r)$ be distributions in $\mathcal{D}$ and let $f(x, r)_n = (f * \delta_n)(x)$, $g(x, r)_n = (g * \delta_n)(x)$. We say that the product $|f| \cdot |g|$ of $f(x, r)$ and $g(x, r)$ exist and is equal to h on the interval (a, b) if

$$\lim_{n\to\infty}\langle f(x, r)_n g(x, r)_n, \phi(x)\rangle = \langle h(x), \phi(x)\rangle$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

The product thus defined is clearly commutative if it exists. The following result shows that the product $f \cdot g$ exists but the product fg defined in Definition 1 does not exist.

$$x^{-r} \cdot \delta^{(r-1)}(x) = \frac{(-1)^r r!}{(2r)!}\delta^{(2r-1)}(x). \tag{1}$$

Later Fisher gave the following non–commutative definition of the product, see [11].

Definition 3. Let $f(x, r)$ and $g(x, r)$ be distributions in $\mathcal{D}'$ and let $g(x, r)_n = (g * \delta_n)(x)$. We say that the product $f \cdot [g]$ of f and g exists and is equal to h on the interval (a, b) if

$$\lim_{n\to\infty}\langle f(x, r)g(x, r)_n, \phi(x)\rangle = \langle h(x), \phi(x)\rangle$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

It is obvious that if the product fg exists by Definition 1 then the product $[f] \cdot [g]$ exists by Definition 2 and $fg = [f] \cdot [g]$. Note also that although the product defined in Definition 2 is always commutative, the product defined in Definition 3 is in general non-commutative, hence in the general $f \cdot [g] \neq [f] \cdot g$. and the results obtained for these products contain constants which depend on the choice of the function ρ.

In general above two products need not necessarily be equal. Several examples were given in [9], [10] and [11] that two products differ. Now we give the following a new definition that these two products are equal under certain conditions.

Definition 4 . Let $f(x, r)$ and $f(x, s)$ be distributions in $\mathcal{D}'$ and let $f(x, r)_n = (f * \delta_n)(t)$. We say that the product is a semigroup product $[f(x, r)] \cdot [f(x, s)]$ (or $f(x, r) \cdot [f(x, s)]$) of $f(x, r)$ and $f(x, s)$ exist and is equal to $f(x, r + s)$ on the interval (a, b) if

$$\lim_{n \to \infty} \langle f(x, r)_n f(x, s)_n, \phi(x) \rangle = \langle f(x, r + s), \phi(x) \rangle$$

or

$$\lim_{n \to \infty} \langle f(x, r) f(x, s)_n, \phi(x) \rangle = \langle f(x, r + s), \phi(x) \rangle$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

Theorem 1. *If the products satisfy the semigroup property then two neutrix products are equal.*

Proof. Let products satisfy the semigroup condition. Then it can be easily seen on using the definitions that

$$\lim_{n \to \infty} \langle f(x, r) f(x, s)_n, \phi(x) \rangle = \langle f(x, r + s), \phi(x) \rangle$$

and

$$\lim_{n \to \infty} \langle f(x, r)_n f(x, s)_n, \phi(x) \rangle = \langle f(x, r + s), \phi(x) \rangle$$

exist respectively for all ϕ in $\mathcal{D}$. Then it follows that

$$\lim_{n\to\infty}\langle f(x, r)f(x, s)_n, \phi(x)\rangle = \langle f(x, r+s), \phi(x)\rangle = \lim_{n\to\infty}\langle f(x, r)_n f(x, s)_n, \phi(x)\rangle$$

for all ϕ in $\mathcal{D}$. Then we have

$$\lim_{n\to\infty}\langle f(x, r)f(x, s)_n - f(x, r)_n f(x, s)_n, \phi(x)\rangle = 0.$$

This proves the theorem.

Corollary 1. *If the product is a semigroup product then it also satisfies the associativity.*

Proof. We will only prove the commutative case, non-commutative case also proved similarly. Let product be a semigroup product and let $f(x, r)$, $f(x, s)$ and $f(x, t)$ be distributions in D'. On using the definition we write that

$$\begin{aligned}[f(x, r)]\cdot[f(x, s)\cdot f(x, t)] &= f(x, r)\cdot \lim_{n\to\infty}\langle f(x, s)_n f(x, t)_n, \phi(x)\rangle \\ &= [f(x, r)]\cdot[f(x, t+s)] \\ &= \langle f(x, r+s+t), \phi\rangle\end{aligned}$$

exists. On the other side we have that

$$\begin{aligned}[f(x, r)\cdot f(x, s)]\cdot[f(x, t)] &= \lim_{n\to\infty}\langle f(x, s)_n f(x, t)_n, \phi(x)\rangle\cdot[f(x, r)] \\ &= [f(x, r+s)]\cdot[f(x, t)] \\ &= \langle f(x, r+s+t), \phi(x)\rangle.\end{aligned}$$

We can easily provide some counter examples that the converse of the above statement, in general, is not correct. For example, it was proved in [4], [11] respectively that the product $\delta^{(s)}\cdot[\delta^{(r)}]$ (and $[\delta^{(s)}]\cdot[\delta^{(r)}]$) exist and

$$[\delta^{(s)}]\cdot[\delta^{(r)}] = \delta^{(s)}\cdot[\delta^{(r)}] = 0. \tag{2}$$

for $s, r = 0, 1, 2, \ldots,$ however it is obvious that the product is not a semigroup product.

Similarly, the distribution $(x+i0)^{-s}$ is defined as follows, see Gel'fand and Shilov [5].

$$(x+i0)^{-s} = x^{-s} + \frac{(-1)^s i\pi}{(s-1)!}(\delta^{(s-1)}(x)$$

then it was proved in [4] that the product is a semigroup product since

$$[(x+i0)^{-s}]\cdot(x+i0)^{-r} = (x+i0)^{-r-s} = [(x+i0)^{-s}]\cdot[(x+i0)^{-r}].$$

References

[1] J. F. Colombeau, Multiplication of distributions, Bull. Amer. Math. Soc. 23(2) (1990), 251-268.

[2] B. Fisher, The product of distributions, Quart. J. Math. Oxford 22(2) (1971), 291-298.

[3] B. Fisher, Biljana Jolevsaka-Tuneska and Adem Kiliçman, On defining the incomplete gamma function, Integral Transform and Special Functions 14(4) (2003), 293-299.

[4] B. Fisher and A. Kiliçman, On the product of the function $x^r \ln(x+i0)$ and the distribution $(x+i0)^{-s}$, Integral Transform and Special Functions 2(4) (1994), 243-252.

[5] I. M. Gel'fand and G. E. Shilov, Generalized Functions, Vol I, Academic Press, 1964.

[6] Y. Hirata and H. Ogata, On the exchange formula for distributions, J. Sci. Hiroshima Univ. Series A 22 (1953), 147-152.

[7] L. Hormander, Fourier integral operators I, Acta Math. 127 (1971), 79-181.

[8] M. Itano, Remarks on the multiplicative product of distributions, Hiroshima Math. J. 6 (1953), 365-375.

[9] A. Kiliçman, On the commutative neutrix product of distributions, Indian J. Pure Appl. Math. 30(8) (1999), 753-762.

[10] A. Kiliçman, A comparison on the commutative convolution of distributions and exchange formula, Czechoslovak Math. J. 51(3) (2001), 463-471.

[11] A. Kiliçman, Distributions Theory and Neutrix Calculus, University Putra Malaysia Press, 2006.

[12] J. Mikusiński and R. Sikorski, The Elementary Theory of Distributions I, Polska Akad. Nauk; Rozprawy Mat., 12, 1957.

[13] J. Mikusiński, On the square of the Dirac delta distribution, Bull. Acad. Pol. Sci. 14(6) (1966), 511-513.

[14] M. Oberguggenberger, Multiplication of distributions and applications to partial differential equations, Pitman Research Notes in Mathematics, Vol. 259, 1992.

[15] L. Schwartz, Theorie des Distributions, Hermann, Paris, 1966.

[16] G. Temple, The theory of generalized functions, Proc. Roy. Soc. Ser. A 228 (1955), 175-190.

[17] Y. Jack Ng and H. Van Dam, An application of Neutrix calculus to quantum field theory, International Journal of Modern Physics A 21(2) (2006), 297-312.

■